Journal of Approximation Theory **101**, 160–163 (1999)
Article ID jath.1999.3362, available online at http://www.idealibrary.com on IDEAL®

NOTE

An Approximation Property and the Sphere of L_1

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Communicated by Frank Deutsch

Received July 6, 1998; accepted in revised form January 15, 1999

A normed linear space, E, has property P if for every $\varepsilon > 0$ there is a $\delta > 0$ such that for all $x \in E$ there is a z, with $|z| \leqslant \varepsilon$, such that $B[0, 1+\delta] \cap B[x, 1] \subseteq B[z, 1]$. This paper contains a simple proof that an infinite dimensional $L_1(X, \Sigma, \mu)$ space does *not* satisfy property P. © 1999 Academic Press

Key Words: l_1^n; geometric property of the sphere of L_1; property P_1.

1. INTRODUCTION

Mach [7, 8] introduced a condition he called property P_1, which is more restrictive than property P (see the definition at the end of the paper) that appears in the theory of simultaneous approximation of functions and in the approximation of compact operators (e.g., [4, 6, 7, 9]). Mach showed that uniformly convex spaces and spaces of continuous functions satisfy the condition. He asked [8] if L_1 spaces also satisfy his property. Kamal [5] gave an intricate computation to show that l_1 does not satisfy Mach's property. By referencing theorems on the attainment of "compact widths" and on embeddings of $L_1[0, 1]$ in $L_1(X, \Sigma, \mu)$, he showed that for (X, Σ, μ) not purely atomic, $L_1(X, \Sigma, \mu)$ also does not satisfy Mach's property.

This note contains an elementary, self-contained proof of Kamal's theorems. That is, we show that infinite dimensional L_1 spaces do not satisfy property P (and hence also do not satisfy Mach's property).

2. PRELIMINARIES

This section contains some immediate consequences of property P. The proof of Lemma 2.2 is included for completeness.

0021-9045/99 $30.00

Let E be a normed linear space. For $x \in E$, the closed sphere of radius r about x is written $B[x, r]$.

DEFINITION. E has property P if for every $\varepsilon > 0$ there is a $\delta > 0$ such that for all $x \in E$ there is a $z \in B[0, \varepsilon]$ such that $B[0, 1+\delta] \cap B[x, 1] \subseteq B[z, 1]$.

Let $\delta_{\varepsilon, E}$ be the supremum of the δ's satisfying the definition of property P.

LEMMA 2.1. *Let E have property P.*

(i) *If $\delta_{\varepsilon, E} > \delta$, then δ satisfies the definition of property P.*

(ii) *$\alpha \leqslant \varepsilon$ implies $\delta_{\alpha, E} \leqslant \delta_{\varepsilon, E}$.*

(iii) *If F is a norm one complemented subspace of E, then F has property P and $\delta_{\varepsilon, E} \leqslant \delta_{\varepsilon, F}$.*

(iv) *If G is isometrically isomorphic to E, then G has property P and $\delta_{\varepsilon, E} = \delta_{\varepsilon, G}$.*

LEMMA 2.2. *If $L_1 = L_1(X, \Sigma, \mu)$ has dimension greater than or equal n, then L_1 contains a norm one complemented subspace that is isometrically isomorphic to l_1^n.*

Proof. Σ contains n disjoint sets, $\{U_i\}_{i=1}^n$, of positive finite measure. For $f \in L_1$, let

$$Lf(x) = \begin{cases} \dfrac{1}{\mu(U_i)} \displaystyle\int_{U_i} f \, d\mu & \text{for} \quad x \in U_i; \\ 0 & \text{otherwise.} \end{cases}$$

Then L is a norm one projection onto the space, F, of functions constant on each U_i and equal 0 on the complement of $\bigcup_{i=1}^n U_i$. Let

$$u_i(x) = \begin{cases} \dfrac{1}{\mu(U_i)} & \text{for} \quad x \in U_i; \\ 0 & \text{otherwise.} \end{cases}$$

Let $\{e_i\}_{i=1}^n$ be the canonical basis for l_1^n (i.e., the ith coordinate of e_i is 1; all others are 0). The linear mapping that takes u_i to e_i is an isometric isomorphism of F onto l_1^n. ∎

3. PROPERTY P AND L_1 SPACES

Throughout this section let $n = 2k + 1$ be an odd integer.

LEMMA 3.1. *Put $x = (1/n, 1/n, ..., 1/n) \in l_1^n$. If $B[0, 1 + (1/n)] \cap B[x, 1] \subseteq B[z, 1]$, then $z = x$.*

Proof. We first show that $z_{k+1} \geqslant 1/n$. Put $u = (0, ..., 0, 2/n, 2/n, ..., 2/n)$ and $v = (2/n, 2/n, ..., 2/n, 0, 0, ..., 0)$, where each of u and v has k coordinates equal 0, and $k+1$ equal $2/n$.

We compute that $|u| = 1 + (1/n) = |v|$, and $|u - x| = 1 = |v - x|$. That is, both u and v are in $B[0, 1 + (1/n)] \cap B[x, 1]$.

If $u \in B[z, 1]$, then $|z_1| + |z_2| + \cdots + |z_k| + ((2/n) - z_{k+1}) + ((2/n) - z_{k+2}) + \cdots + ((2/n) - z_n) \leqslant 1$, and $\pm z_1 \pm z_2 \cdots \pm z_k + z_{k+1} + z_{k+2} + \cdots + z_n \geqslant 1/n$.

Similarly $v \in B[z, 1]$ implies, $z_1 + z_2 \cdots + z_k + z_{k+1} \pm z_{k+2} + \cdots \pm z_n \geqslant 1/n$.

Choosing all minus signs and adding the last two inequalities yields $z_{k+1} \geqslant 1/n$. The same reasoning shows each $z_i \geqslant 1/n$.

Since $0 \in B[0, 1 + (1/n)] \cap B[x, 1]$, we must have that $|z - 0| \leqslant 1$, and so each $z_i = 1/n$. ∎

COROLLARY 3.2. $\delta_{\varepsilon, l_1^n} < 1/n$ *for all* $0 < \varepsilon < 1$.

THEOREM 3.3. *If* $L_1 = L_1(X, \Sigma, \mu)$ *is infinite dimensional, then* L_1 *does not have property P.*

Proof. Fix $1 > \varepsilon > 0$. From Lemma 2.1, 2.2, and Corollary 3.2 for all odd integers n, we have $\delta_{\varepsilon, L_1} \leqslant \delta_{\varepsilon, l_1^n} < 1/n$. So $\delta_{\varepsilon, L_1} = 0$, which is not compatible with property P. ∎

Comment. A normed linear space, E, has *Mach's property* (or property P_1) if for every $\varepsilon > 0$ and each $r > 0$ there is a $\delta > 0$ such that for all x and y in E and all $0 < \theta < \delta$ there is a $z \in B[x, \varepsilon]$ such that $B(x, r + \delta) \cap B(y, r + \theta) \subseteq B(z, r + \theta)$. Here $B(\cdot, \cdot)$ represents an open sphere. Property P_1 is the property referred to in the Introduction. If E has Mach's property, then it also satisfies property P.

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